Modelling of free boundary phenomena in thin films mechanics: a bifluid approach

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Fed3G - MaiMoSiNE - "Mouillage, démouillage, imprégnation"
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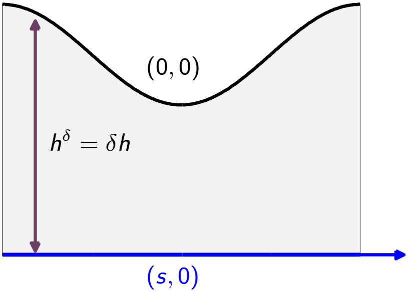
- **References:** Reynolds 1886, Cimatti '78, Bayada & Chambat '86, Nazarov '90

- **Stokes equations:**

$$\begin{aligned} -\mu\Delta\mathbf{u}^\delta + \nabla p^\delta &= 0 \\ \operatorname{div}\mathbf{u}^\delta &= 0 \end{aligned}$$

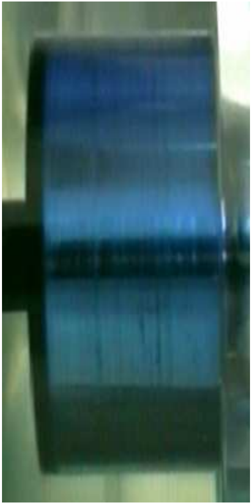
- **Thin film ($\delta \ll 1$):**

$$\begin{aligned} \delta^2 p^\delta(x,z) &\to p^\star(x) \\ \mathbf{u}^\delta(x,z) &\to \mathbf{u}^\star(x,z) \end{aligned}$$



- **Reynolds equation:**

$$-\partial_x\left(\frac{h^3}{12\mu}\partial_x p^\star\right) = -\frac{s}{2}\partial_x h, \quad \mu\partial_{zz}u_1^\star = \partial_x p^\star, \quad u_2^\star = 0$$

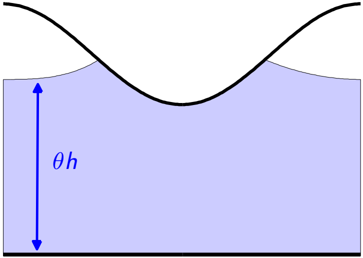


Mechanics: Floberg, Jakobsson & Olsson '57, Elrod & Adams '78.

Mathematical analysis: Bayada & Chambat '82, Alvarez & Carrillo '90, Carrillo & Chipot '93, Vázquez '94, Alvarez & Oujja '03

$$-\partial_x \left(\frac{h^3}{12\mu} \partial_x p \right) = -\frac{s}{2} \partial_x (\theta h),$$
$$p \geq 0, \qquad H(p) \leq \theta \leq 1$$

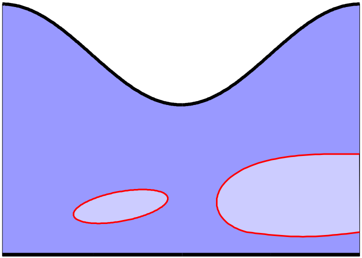
- satured area : $p \geq 0, \quad \theta = 1$
- cavitated area : $p = 0, \quad 0 \leq \theta < 1$



Consider a flow of two immiscible fluids with viscosities μ_ℓ and $\mu_g < \mu_\ell$

Multifluid Stokes equations

$$\begin{aligned} -\mathrm{div}(2\mu\mathbb{D}(\mathbf{u}) - p\,\mathrm{I}) &= 0 \\ \mathrm{div}\,\mathbf{u} &= 0 \\ \partial_t\mu + \mathrm{div}(\mu\,\mathbf{u}) &= 0 \end{aligned}$$



Theorem 1 (Nouri, Poupaud & Demay '97)

Under suitable initial and boundary conditions, in particular

$$\mu^0 \in \{\mu_\ell, \mu_g\},$$

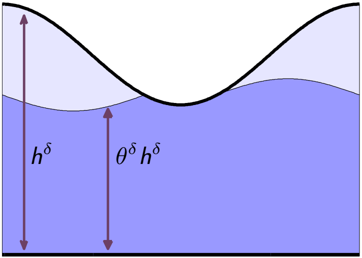
the multifluid Stokes problem admits a solution $(\mathbf{u}, p, \mu)$. *Moreover*

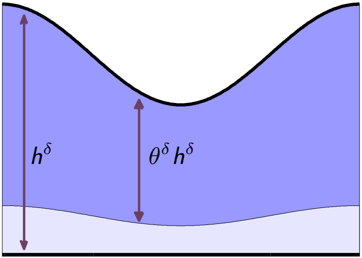
$$\mu \in \{\mu_\ell, \mu_g\}.$$

Graph assumption on the free boundary: introducing the saturation

$$\theta^\delta(x) = \frac{1}{h^\delta(x)} \int_0^{h^\delta(x)} \frac{\mu^\delta(x,z) - \mu_\ell}{\mu_g - \mu_\ell} \, \mathrm{d}z$$

Thin film assumption for the bifluid model: asymptotic behaviour of the multifluid Stokes system $\delta \to 0$





Proposition 1 (Paoli '03)

Let $\varepsilon := \mu_g/\mu_\ell$. The bifluid thin film model is described by:

- *Reynolds equation for the pressure $x \mapsto p(\cdot, x)$*

$$-\partial_x \left(\mathcal{A}_\varepsilon(\theta) \frac{h^3}{12\mu_\ell} \partial_x p \right) = -\partial_x \left(\mathcal{B}_\varepsilon(\theta) \frac{sh}{2} \right).$$

- *Buckley-Leverett equation for the liquid saturation $x \mapsto \theta(\cdot, x)$*

$$\partial_t (\theta h) + \partial_x (Q_{in} \mathcal{F}_\varepsilon(\theta) + sh\, \mathcal{G}_\varepsilon(\theta)) = 0.$$

Questions arising from this bifluid model:

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- asymptotic behaviour with respect to ε
- comparison between Elrod-Adams and bifluid models

Outline

Introduction: cavitation in lubrication and bifluid models

1. Scalar conservation laws on bounded domains

2. Long-time behaviour of the entropy solution (w J. Vovelle)

3. Scalar conservation laws with discontinuous flux (w J. Vovelle)

4. Comparison between Elrod-Adams and bifluid models (w G. Bayada, C. Vázquez)

Conclusion

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1. Scalar conservation laws on bounded domains.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$:

$$\begin{array}{rcll} \partial_t u + \operatorname{div}_x \left(\mathcal{F}(\cdot,\cdot,u)\right) + \mathcal{G}(\cdot,\cdot,u) & = & 0, & \text{in } Q_T := (0,T)\times\Omega, \\ u(0,\cdot) & = & u^0, & \text{on } \Omega, \\ u & = & u^D, & \text{on } \Sigma_T := (0,T)\times\partial\Omega, \end{array}$$

Assumptions:

i. $\mathcal{F}$ and $\mathcal{G}$ are regular functions,

ii. $(\mathrm{div}_x\mathcal{F} + \mathcal{G})(\cdot,\cdot,a) \leq 0$ and $(\mathrm{div}_x\mathcal{F} + \mathcal{G})(\cdot,\cdot,b) \geq 0$,

iii. $(u^0, u^D) \in L^\infty(\Omega; [a,b]) \times L^\infty(\Sigma_T; [a,b])$.

References

- Kruzkov '71 (unbounded domains)
- Bardos, Le Roux & Nédélec '79 (bounded domain and regular data)
- Otto '96 (L^∞ data with autonomous fluxes and no source term)

Definition. Suppose that the data are BV. A function $u \in BV(Q_T)$ is a weak entropy solution if

$$\int_{Q_T} \Big\{ \Big|u-\kappa\Big| \partial_t \varphi + (\mathrm{sgn}(u-\kappa)(\mathcal{F}(\cdot,\cdot,u)) - \mathcal{F}(\cdot,\cdot,\kappa))) \cdot \nabla\varphi - \mathrm{sgn}(u-\kappa)\,(\mathrm{div}\mathcal{F}(\cdot,\cdot,\kappa) + \mathcal{G}(\cdot,\cdot,u))\ \varphi \Big\} + \int_\Omega \Big|u^0-\kappa\Big|\ \varphi(0,\cdot) - \int_{\Sigma_T} \mathrm{sgn}(u^D-\kappa) \Big\{ \mathcal{F}(\cdot,\cdot,u) - \mathcal{F}(\cdot,\cdot,\kappa) \Big\} \cdot \mathbf{n}\ \varphi \geq 0,$$

for every $\phi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$, $\phi \geq 0$, for every $\kappa \in \mathbb{R}$.

Theorem 2 (Bardos, Le Roux & Nédélec '79)

Suppose that the data are BV. There exists a unique entropy solution.

1. Scalar conservation laws on bounded domains.

- **Boundary condition given** by Bardos, Le Roux and Nédélec:

$$\min_{\kappa \in \mathcal{I}[\gamma u, u^D]} \operatorname{sgn}(\gamma u - u^D)(\mathcal{F}(\cdot,\cdot,\gamma u) - \mathcal{F}(\cdot,\cdot,\kappa)) \cdot \mathbf{n} = 0.$$

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- **Monotonicity assumption.** Assume furthermore that

$$u \mapsto \mathcal{F}(\cdot,\cdot,u) \cdot \mathbf{n} \text{ is a monotone function.}$$

Then, the boundary condition reads:

$$\begin{array}{ll} u = u^D & \text{on } \{(t,r) \in \Sigma_T,\ \partial_u \mathcal{F}(t,r,u) \cdot \mathbf{n}(r) < 0\} =: \Sigma_T^{\text{in}} \\ \text{nothing} & \text{on } \{(t,r) \in \Sigma_T,\ \partial_u \mathcal{F}(t,r,u) \cdot \mathbf{n}(r) \geq 0\} =: \Sigma_T^{\text{out}} \end{array}$$

Definition. Suppose that the data are L^∞. A function $u \in L^\infty(Q_T)$ is an entropy solution if

$$\int_{Q_T} \Big\{ (u-\kappa)^{\pm}\, \partial_t \varphi + \big(\mathrm{sgn}_{\pm}(u-\kappa)(\mathcal{F}(\cdot,\cdot,u) - \mathcal{F}(\cdot,\cdot,\kappa))\big) \cdot \nabla\varphi$$

$$-\mathrm{sgn}_{\pm}(u-\kappa)\Big(\mathrm{div}_x \mathcal{F}(\cdot,\cdot,\kappa) + \mathcal{G}(\cdot,\cdot,u)\Big)\, \varphi \Big\}$$

$$+ \int_{\Omega} (u^0-\kappa)^{\pm}\, \varphi(0,\cdot) + \mathcal{L}_{[\mathcal{F}]} \int_{\Sigma_T} \left(u^D - \kappa\right)^{\pm} \varphi \geq 0,$$

for every $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^d)$, $\phi \geq 0$, for every $\kappa \in \mathbb{R}$.

Theorem 3 (Martin '07)

Suppose that the data are L^∞. There exists a unique entropy solution.

- entropy formulation: Kruzkov *semi*-entropy / flux ;
- existence by parabolic regularization ;
- uniqueness by the doubling variable method.

1. Scalar conservation laws on bounded domains.

Finite volume scheme. Define the following discrete quantities:

$$\forall K \in \mathcal{T},\ u_K^0 = \frac{1}{|K|}\int_K u^0 \qquad \text{(volumic average on cells inside the domain)}$$
$$\forall \sigma \in \mathcal{E}_B,\ u_\sigma^b = \frac{1}{|\sigma|}\int_{\mathcal{E}_B} u^D \qquad \text{(lineic average on the boundary edges)}$$

Iterations in time:

$$\forall K \in \mathcal{T}, \quad \frac{u_K^{n+1} - u_K^n}{k} + \frac{k}{|K|}\sum_{\sigma \in \mathcal{E}_K} F_{K\sigma}(u_K^n, u_{K,\sigma}^n) = 0, \quad n \in \mathbb{N}.$$

with suitable assumptions on the numerical flux $F_{K\sigma}$ (monotonicity, consistency, conservativity, regularity) and

$$u_{K,\sigma}^n = \begin{cases} u_L^n, & \text{if } \sigma = K|L \quad \text{(edge inside the domain)} \\ u_\sigma^b, & \text{if } \sigma \in \mathcal{E}_b \quad \text{(edge of the boundary)} \end{cases}$$

Theorem 4 (Vovelle '02)

Under a CFL condition, the numerical solution converges to the unique entropy solution of the problem in $L^p(Q_T)$, $1 \le p < +\infty$.

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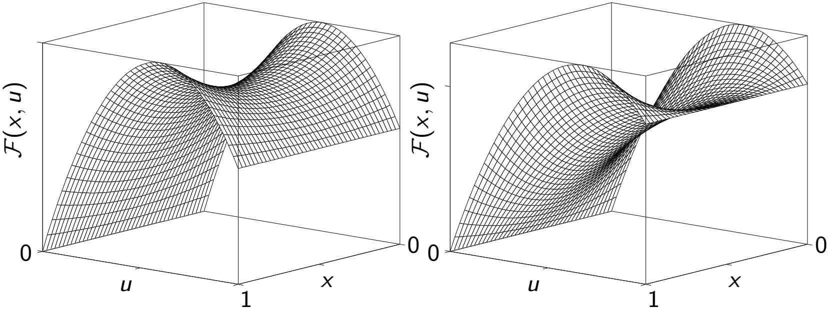
- **Scalar conservation law on bounded domain with non-autonomous flux:**

$$\begin{aligned} \partial_t u(t,x) + \partial_x\left(\mathcal{F}(x,u(t,x))\right) &= 0, && (t,x)\in Q_T := (0,T)\times(0,1),\\ u(0,x) &= u^0(x), && x\in(0,1),\\ u(t,r) &= \bar{u}\in\mathbb{R}, && (t,r)\in(0,T)\times\partial(0,1). \end{aligned}$$

- **Structural assumptions:** $\mathcal{F}_{\mathrm{crit.}} := \min\limits_{x\in[0,1]} \max\limits_{u\in[0,1]} \{\mathcal{F}(x,u)\}$



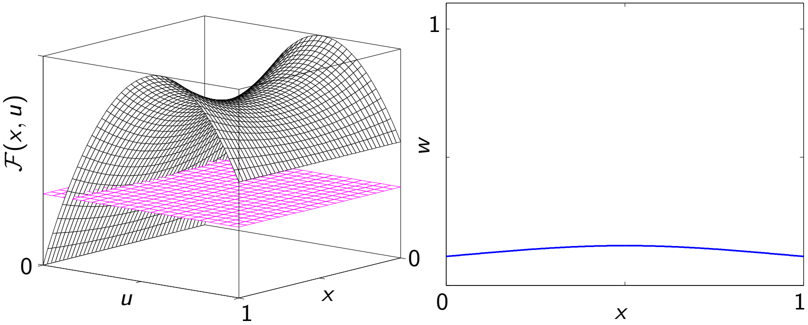
- **Model case:** $\mathcal{F}(x,u) = Q\,u + H(x)\,u\,(1-u).$

■ **Stationary states:** we focus on functions $w \in \mathcal{C}([0,1];[0,1])$ such that

$$\mathcal{F}(x, w(x)) = f^0 \in \mathbb{R}, \quad x \in [0,1]$$

Depending on the data, the equation admits

One solution | Two solutions $\{\lambda, \mu\}$ with $\lambda \leq \mu$ | No solution.



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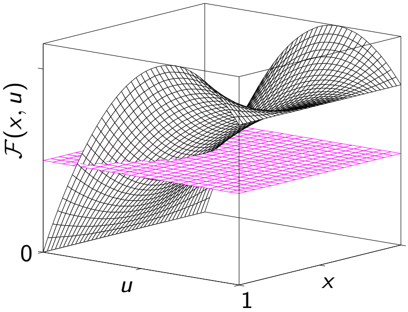

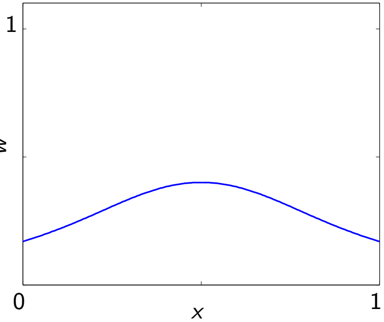

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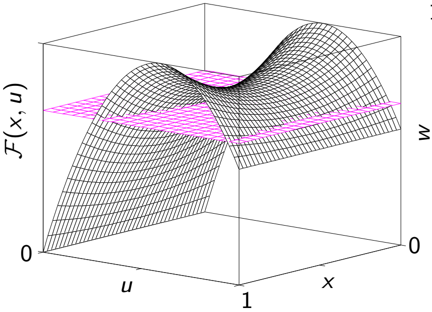

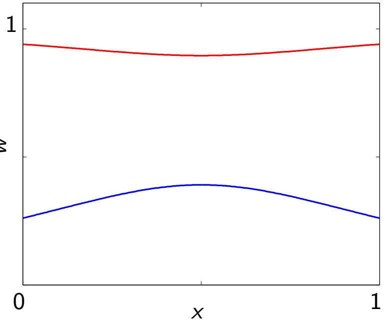

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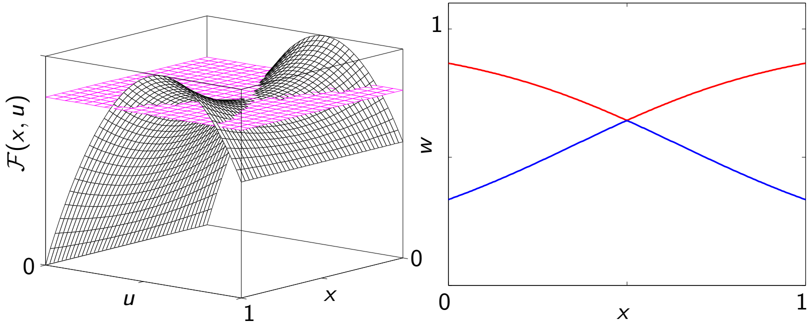

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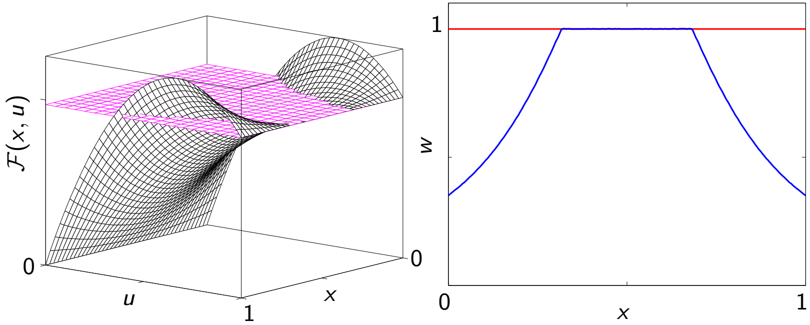

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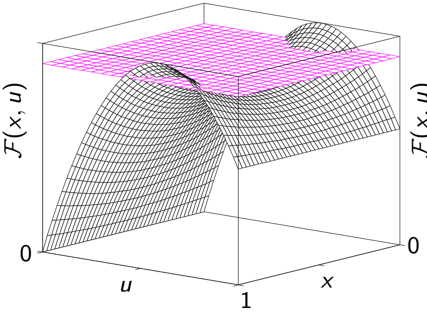

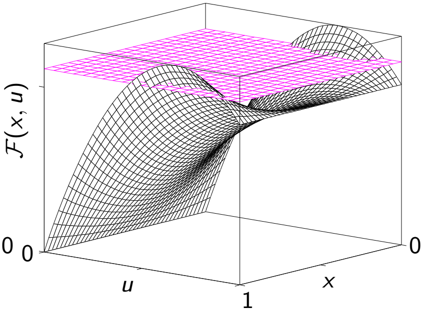

■ Convergence of the solution:

Theorem 5 (Martin & Vovelle '07)

- *For all* $u_0 \in L^\infty(0,1;[0,1])$, $\bar{u} \in [0,1]$, *the entropy solution converges to a stationary state including possible discontinuities (stationary entropy shocks), the constant* f^0 *being fixed by*

$$f^0 = \min(\mathcal{F}(0,\bar{u}), \mathcal{F}_{\mathrm{crit.}}).$$

- *Stationary entropy shocks are possible if and only if* $\mathcal{F}(0,\bar{u}) = \mathcal{F}_{\mathrm{crit.}}$.
  - ◦ *if* $u_0 \leq \lambda$ *(resp.* $u_0 \geq \mu$*), then* u *converges to* λ *(resp.* μ*),*
  - ◦ *if* $\lambda \leq u_0 \leq \mu$, *then there exists a stationary shock.*

Sketch of the proof: semi-group analysis as done by Freistühler & Serre '01.

2. Long-time behaviour of the entropy solution.

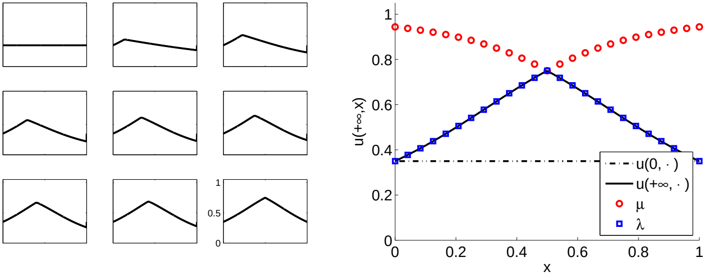

Figure: Critical regime: $\mathcal{F}(0, \bar{u}) = \mathcal{F}_{\text{crit.}}$.

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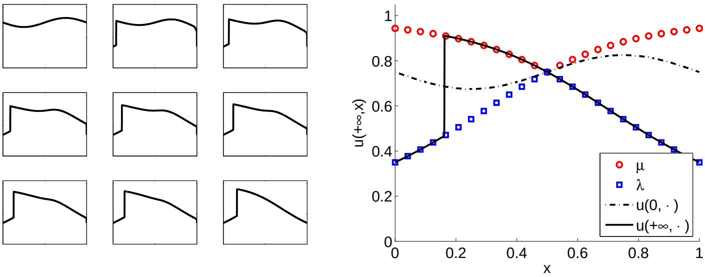


Figure: Critical regime: $\mathcal{F}(0, \bar{u}) = \mathcal{F}_{\text{crit.}}$.

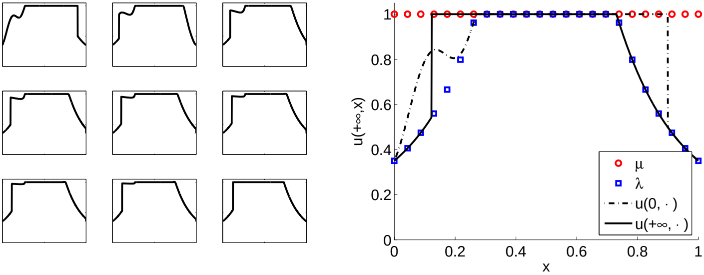


Figure: Critical regime: $\mathcal{F}(0, \bar{u}) = \mathcal{F}_{\text{crit.}}$.

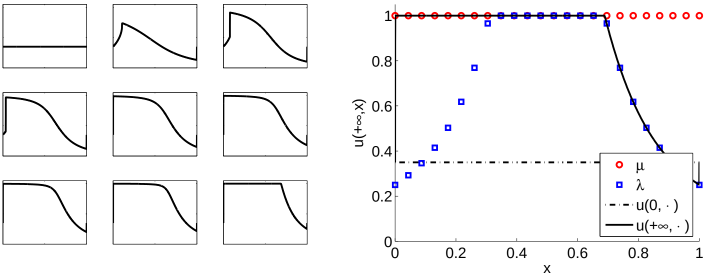


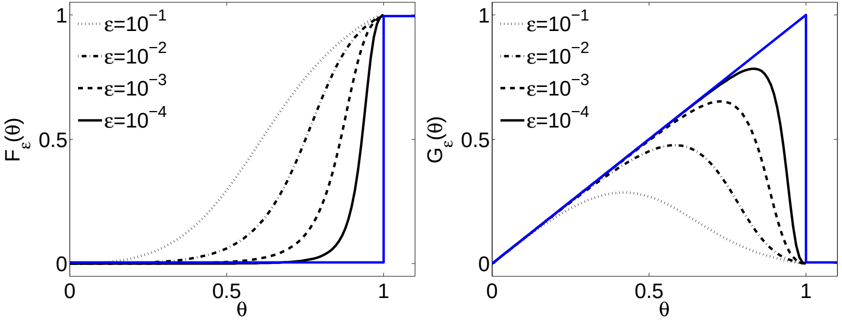
Figure: Non-critical regime: $\mathcal{F}(0,\bar{u}) > \mathcal{F}_{\text{crit.}}$.

Outline

Buckley-Leverett equation in the bifluid model:

$$\partial_t(\theta h) + \partial_x \left(Q_{in} \mathcal{F}_\varepsilon(\theta) + sh\, \mathcal{G}_\varepsilon(\theta) \right) = 0$$

Asymptotic behaviour: $\mathcal{F}_\varepsilon(\theta) \xrightarrow[\varepsilon \to 0]{} H(\theta - 1)$ and $\mathcal{G}_\varepsilon(\theta) \xrightarrow[\varepsilon \to 0]{} \theta\, H(1 - \theta)$



Question: investigation of scalar conservation laws with discontinuous flux.

■ **Mathematical formulation:**

$$\mathrm{P}(u^0, f)\left\{\begin{array}{rcll} \partial_t u + \mathrm{div}(\mathcal{F}(u)) & = & f, & \text{in } Q_T, \\ u(0,\cdot) & = & u^0, & \text{on } \Omega, \\ u & = & 0, & \text{on } (0,T)\times\partial\Omega. \end{array}\right.$$

where the flux satisifies $\mathcal{F}(\alpha^+) \neq \mathcal{F}(\alpha^-)$.

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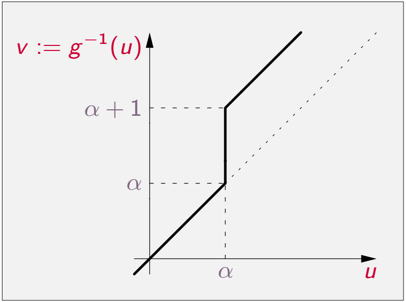
where the flux satisifies $\mathcal{F}(\alpha^+) \neq \mathcal{F}(\alpha^-)$.

Define

$$v := u + \theta_\alpha(u)$$

such that

- if $u < \alpha$, $\theta_\alpha(u) = 0$
- if $u = \alpha$, $\theta_\alpha(u) \in [0,1]$
- if $u > \alpha$, $\theta_\alpha(u) = 1$



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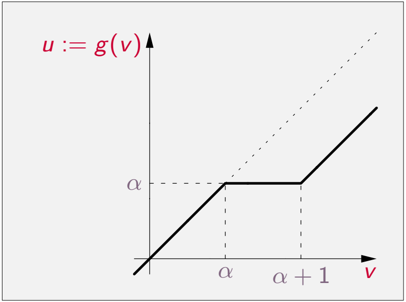
$$v := u + \theta_\alpha(u)$$

such that

- if $u < \alpha$, $\quad \theta_\alpha(u) = 0$
- if $u = \alpha$, $\quad \theta_\alpha(u) \in [0,1]$
- if $u > \alpha$, $\quad \theta_\alpha(u) = 1$



■ **Mathematical formulation:**

$$\mathrm{P}(u^0, f) \begin{cases} \partial_t u + \mathrm{div}(\mathcal{F}(u)) &= f, \quad \text{in } Q_T, \\ u(0,\cdot) &= u^0, \quad \text{on } \Omega, \\ u &= 0, \quad \text{on } (0,T)\times\partial\Omega. \end{cases}$$

where the flux satisifies $\mathcal{F}(\alpha^+) \neq \mathcal{F}(\alpha^-)$.

■ **Reformulation of the problem** (Carrillo '03):

$$\widetilde{\mathrm{P}}(u^0, f) \begin{cases} \partial_t(g(v)) + \mathrm{div}(\mathcal{G}(v)) &= f, \quad \text{sur } Q_T, \\ g(v)(0,\cdot) &= u^0, \quad \text{sur } \Omega, \\ g(v) &= 0, \quad \text{sur } (0,T)\times\partial\Omega. \end{cases}$$

■ **Consequences:**

+ Lipschitz regularity for $\mathcal{G}$;
– g^{-1} is non-univoque ;
– numerical schemes.

Implicit finite volume scheme based on $\widetilde{\mathrm{P}}(u^0, f)$:

$$\forall K \in \mathcal{T},\ u_K^0 = \frac{1}{|K|}\int_K u^0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad f_K^n = \frac{1}{k|K|}\int_{nk}^{(n+1)k}\int_K f(t,\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$
$$\forall K \in \mathcal{T},\ g(v_K^{n+1}) = u_K^n - \frac{k}{|K|}\sum_{L\in\partial K}\widetilde{F}_{KL}^n(v_K^{n+1}, v_L^{n+1}) + k\, f_K^n, \quad n \in \mathbb{N},$$
$$\forall K \in \mathcal{T},\ u_K^{n+1} = g(v_K^{n+1}), \quad n \in \mathbb{N}.$$

Theorem 6 (Martin & Vovelle '07)

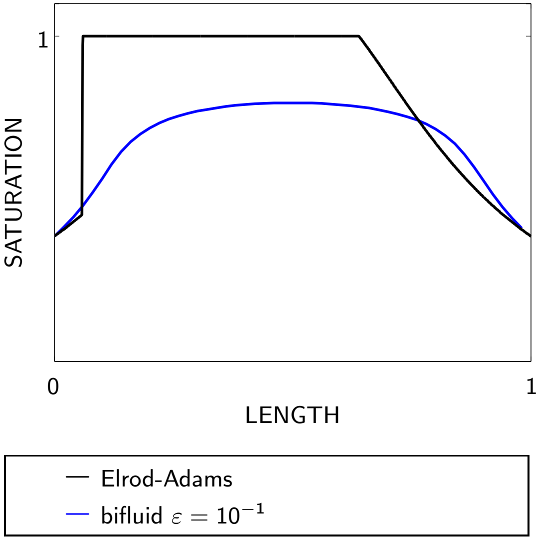
The numerical scheme admits a unique solution, which converges to the unique entropy solution of $\widetilde{\mathrm{P}}(u^0, f)$ *in* $L^p(Q_T)$, $1 \leq p < +\infty$.

Proof:

- existence: topological degree in finite dimension ;
- uniqueness: contraction principle ;
- convergence: notion of *entropy process solution* (Eymard, Gallouët & Herbin '00) which is equivalent to the notion of *measured valued solution* (Szepessy '91)

Outline

4. Comparison between Elrod-Adams and bifluid models.



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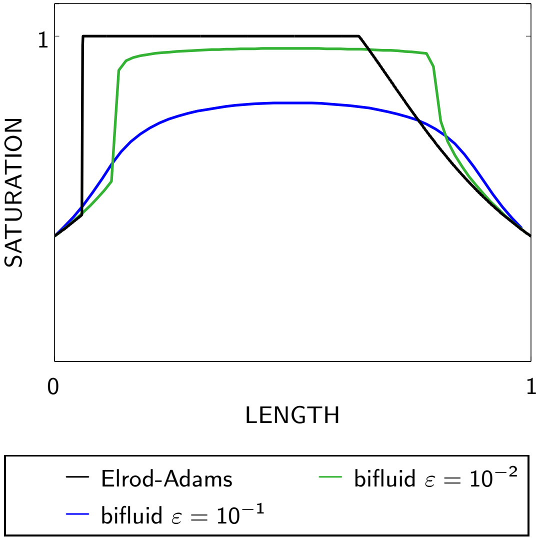

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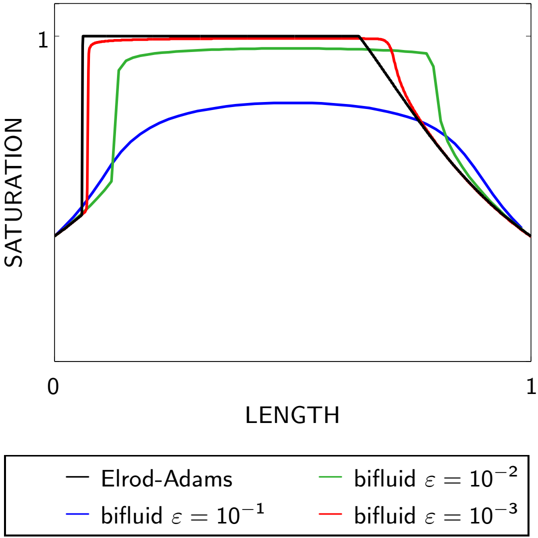

Outline

Conclusion.

- Elrod-Adams and bifluid models
- *Uniqueness* of the Elrod-Adams solution and *non-uniqueness* of the stationary solution for scalar conservation laws
- Boundary conditions and triple point
- Other bifluid models (Bayada, Chupin & Grec '10, ...)